STABILITY OF MEAN CONVEX CONES UNDER MEAN CURVATURE FLOW

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Abstract. We consider graphical solutions to mean curvature flow and obtain a stability result for homothetically expanding solutions coming out of cones of positive mean curvature: If another solution is initially close to the cone at infinity, then the difference to the homothetically expanding solution becomes small for large times. The proof involves the construction of appropriate barriers.

1. Introduction

We study solutions to graphical mean curvature flow

$$\dot{u} = \sqrt{1+|Du|^2}\ \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) \equiv \sqrt{1+|Du|^2}\ H[u] \tag{1.1}$$

for functions $u \in C^\infty_{loc}(\mathbb{R}^n \times (0,\infty)) \cap C^0_{loc}(\mathbb{R}^n \times [0,\infty))$. This equation is known to have a solution for initial data $u(\cdot,0) \in C^0_{loc}(\mathbb{R}^n)$. Let $k : \mathbb{R}^n \to \mathbb{R}$ be smooth outside the origin and positive homogeneous of degree one. Then $\operatorname{graph} k \subset \mathbb{R}^{n+1}$ is a cone. The unique solution U to (1.1) with $U(\cdot,0) = k$ is homothetically expanding, which implies that for any t_1, $t_2 > 0$, $\operatorname{graph} U(\cdot,t_1)$ and $\operatorname{graph} U(\cdot,t_2)$ differ only by a homothety. Hence U fulfills

$$U(x,t) = \sqrt{2nt}\ U\left(\frac{x}{\sqrt{2nt}}, \frac{1}{2n}\right) = \sqrt{t}\ U\left(\frac{x}{\sqrt{t}}, 1\right). \tag{1.2}$$

We refer to [10, 17] for details.

In this paper, we are concerned with solutions u to (1.1) such that $u(\cdot,0) = u_0$ is *close to k at infinity,*

$$\sup_{\mathbb{R}^n \setminus B_r(0)} |u_0 - k| \to 0 \quad \text{as} \quad r \to \infty. \tag{1.3}$$

In this situation, we study stability of U under mean curvature flow, see our main result, Theorem 1.3. It implies in particular the following

Theorem 1.1. *Let k, U, u_0 and u be as above. Assume that* $\operatorname{graph} k$ *is contained in a half-space and has positive mean curvature outside the origin, $H[k] > 0$. Then*

$$\sup_{\mathbb{R}^n} |u(\cdot,t) - U(\cdot,t)| \to 0 \quad \textit{as} \quad t \to \infty.$$

Date: November 2008.
2000 *Mathematics Subject Classification.* 53C44, 35B35.
Key words and phrases. Mean curvature flow, cone, stability.
Supported by ANU, ARC, AvH foundation and DFG.

The results of this paper hold for the following class of mean convex cones:

Definition 1.2. A function $k : \mathbb{R}^n \to \mathbb{R}$ is said to be of class $\mathcal{K}$ if the following conditions are fulfilled:

(i) k is positive homogeneous of degree one;
(ii) k is smooth outside the origin;
(iii) k has non-negative mean curvature $H[k]$ outside the origin;
(iv) there exists a linear function l such that $k \geq l$;
(v) if $n \geq 3$, at points $p = (\hat{p}, p^{n+1}) \neq 0$ in $\operatorname{graph} k$ where $H[k](p) = 0$, we require that the second fundamental form A of $\operatorname{graph} k$ fulfills

$$|A|^2(p) < \left(\tfrac{n-2}{2}\right)^2 |p|^{-2}.$$

We will refer to both the function k and the hypersurface $\operatorname{graph} k$ as cones.

Our main theorem is

Theorem 1.3. *Let U be the homothetically expanding solution to Equation* (1.1) *with $U(\cdot, 0) = k$, where k is a cone of class $\mathcal{K}$. Suppose that $u \in C^\infty_{loc}(\mathbb{R}^n \times (0, \infty)) \cap C^0_{loc}(\mathbb{R}^n \times [0, \infty))$ solves* (1.1) *with initial data $u_0 \in C^0_{loc}(\mathbb{R}^n)$ approaching k at infinity by fulfilling* (1.3)*. Then*

$$u(\cdot, t) - U(\cdot, t) \to 0 \quad as \quad t \to \infty,$$

uniformly in $C^k(\mathbb{R}^n)$ for every $k \in \mathbb{N}$.

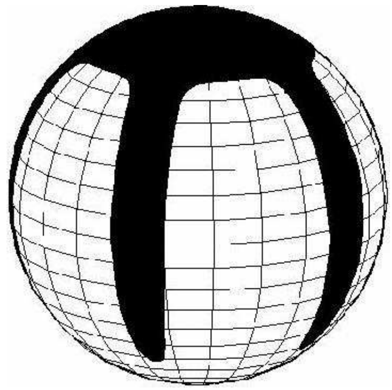
The cones k considered in Theorem 1.3 are such that the homothetically expanding solutions U studied in [10, 17] fulfill $U(\cdot, t) \geq k$ for every $t \geq 0$. It is clear that this requires $H[k] \geq 0$ outside the origin. We also want to impose a non-negativity condition on $H[k]$ at the origin. The definition of viscosity solutions suggests a requirement that $\operatorname{graph} k$ is contained in a half-space; this is implied by Condition (iv) in Definition 1.2. Note that this condition may be violated even if $H[k] \geq 0$ outside the origin. A counterexample is the higher dimensional analogue of what we try to illustrate in the picture. The black region is given by

$$\{(x, z) \in \mathbb{R}^n \times \mathbb{R} : z > k(x)\} \cap \mathbb{S}^n.$$

It is a starshaped subset of the sphere with respect to the north pole as its boundary is given by $\operatorname{graph} k$, intersected with $\mathbb{S}^n$. In higher dimensions, the black region is constructed as follows. Attach sets of the form $B_r \times [0, 1]$ to a geodesic ball around the origin. We smooth the resulting set, especially near $B_r \times \{0\}$ and $B_r \times \{1\}$. As there are hypersurfaces of positive mean curvature that contain "necks" [14], we can ensure that the boundary of the constructed set has positive mean curvature. If we attach enough sets of the form $B_r \times [0, 1]$ that extend over the equator, it is easy to see that the resulting set is not contained in a half-space any more. Hence the corresponding assumption on k is not redundant.

Homogeneous minimal cones fulfill a linearized stability condition if and only if $|A|^2(p) \leq \left(\tfrac{n-2}{2}\right)^2 |p|^{-2}$, see [4, Example 4.7] for details. Hence it is not surprising that we have to impose such a bound at those points, where the mean curvature H vanishes.

If k is convex, Condition (v) in Definition 1.2 always holds.



Our proof of Theorem 1.3 also works if $k \in C^2_{loc}$ away from the origin. We conjecture that the result extends also to continuous cones, but expect that this will be quite technical.

The existence of homothetically expanding solutions to mean curvature flow starting from a cone was first examined by K. Ecker and G. Huisken in [10] and further investigated by N. Stavrou [17]. For graphical initial data, the existence of solutions to (1.1) is studied by T. Colding and W. Minicozzi and others in [7, 8, 10]. After appropriate rescaling, solutions which deviate initially sublinearly from a cone, converge for large times to the homothetically expanding solution. This is proved in [10, 17]. The present paper addresses the corresponding convergence result without rescaling. Such questions have also been addressed in [2, 5, 6, 15].

As in those papers addressing stability without rescaling, we have to impose a decay condition, in our case (1.3). Boundedness of the perturbation does at most imply subsequential convergence to a homothetically expanding solution as appropriate initial oscillations of $u_0 - k$ in space may yield oscillations of $u(0,t) - U(0,t)$ in time. Note however, that we do not have to impose a decay rate in (1.3). Moreover, using the clearing out lemma we can weaken (1.3) by allowing small additional BV-perturbations of k_0. Compare this with [15, Theorem 1.6].

The idea of the proof of Theorem 1.3 and the organization of the rest of the paper are as follows. Here we will only sketch the proof in the case that $U(\cdot,t) > k$ for all $t > 0$. Otherwise, k is linear. We address this special situation in Appendix A.

If $u_0 \geq k$, we can use the homothetically expanding solutions U as barriers. For arbitrary $\varepsilon > 0$ and $T > 0$ chosen appropriately, we have

$$U(x,t) - \varepsilon \leq u(x,t) \leq U(x,t+T) + \varepsilon$$

for all (x,t). As $U(x,t+T) - U(x,t)$ converges to zero as $t \to \infty$ and $\varepsilon > 0$ is arbitrary, convergence follows. We will discuss that in detail in Section 2.

If $u_0 < k$ somewhere, we construct barriers that show that for every $\delta > 0$, there exists $t_\delta > 0$ such that $u(\cdot,t_\delta) \geq k - \delta$. Then the above argument can be applied with $\varepsilon = 2\delta$ and

$$U(x,t) - \varepsilon \leq u(x,t+t_\delta) \leq U(x,t+T) + \varepsilon.$$

The barrier construction and details for that part are to be found in Section 3. In the case of convex cones, we can use hyperplanes as considered in Appendix A to

ensure the existence of such a time t_δ for every $\delta > 0$. The rest of the argument is similar to the one given above. In Section 4 we explain how to weaken the decay condition to allow additional decaying perturbations in BV_{loc}.

In the appendices, we study stability of hyperplanes and the uniform convergence for a family of perturbations.

We want to thank Ben Andrews, Gerhard Huisken and Felix Schulze for discussions. We gratefully acknowledge support from the Alexander von Humboldt foundation, the Australian National University, the Australian Research Council and the Deutsche Forschungsgemeinschaft.

2. One-Sided Perturbations

Let us first show that the conditions imposed on k ensure that the homothetic solution always lies above the cone.

Lemma 2.1. *Assume that $k : \mathbb{R}^n \to \mathbb{R}$ satisfies conditions* (i)–(iv) *of Definition 1.2. Let U be the homothetically expanding solution to* (1.1) *with $U(\cdot, 0) = k$. Then $U(\cdot, t) \ge k$ for all $t \ge 0$.*

Proof. Mean curvature flow of smooth compact manifolds preserves the condition $H \ge 0$. Here, however, we consider a noncompact manifold and the existence proof in [10] involves a mollification of the initial data. This mollification, however, might destroy the condition $H \ge 0$. Hence the result does not seem to be trivial.

According to Appendix A, we have $U(0,t) \ge k(0) = 0$ for all $t \ge 0$. We may assume that k is not a linear function for otherwise $U(\cdot, t) = k$ for all t. As $U(\cdot, t)$ is smooth for $t > 0$ but k is singular at the origin, we deduce that $U(0,t) > 0$ and $H[U](0,t) > 0$ for $t > 0$. Both inequalities extend to a possibly time-dependent neighborhood of the origin. Near spatial infinity, comparison with spheres shows that $\sup\limits_{\mathbb{R}^n \setminus B_r(0)} |U(\cdot, t) - k| \to 0$ uniformly as $r \to \infty$ for t in a bounded time interval. Fix $\varepsilon > 0$ and assume that there exists (x_0, t_0) such that $U(x_0, t_0) - k(x_0) \le -\varepsilon$. The behavior of U at spatial infinity ensures that a negative infimum of $U(\cdot, t) - k$ is attained. Hence we may assume that (x_0, t_0) is such that $t_0 > 0$ is minimal with $U(x_0, t_0) - k(x_0) \le -\varepsilon$. The considerations above imply that $|x_0| > 0$. As $H[k](x_0) \ge 0$, this contradicts the strong maximum principle applied to U and k. The claim follows. □

Corollary 2.2. *Let $k : \mathbb{R}^n \to \mathbb{R}$ be as in Lemma 2.1. If k is not a linear function, then $U(\cdot, t) > k$ for $t > 0$.*

Proof. According to the proof of Lemma 2.1, we have $U(\cdot, t) \ge 0$ and $U(0,t) > 0$ for $t > 0$. Hence the strong maximum principle implies that $U(\cdot, t) > 0$ for $t > 0$. The claim follows. □

In fact $\dot{U} > 0$, or equivalently $H > 0$, for $t > 0$. This follows by adapting techniques of B. White [18]. We do not need this result for proving our Main Theorem 1.3. Therefore we will only sketch the proof.

Lemma 2.3. *Let U be as in Lemma 2.2. Then $\dot{U}(\cdot, t) > 0$ for $t > 0$.*

Sketch of proof: Recall that [18, Theorem 3.1] asserts for compact mean convex level set solutions $F_t(K)$ to mean curvature flow that $F_{t+h}(K) \subset \text{interior } F_t(K)$ for every $t, h > 0$.

We proceed as in the proof of [18, Theorem 3.1], with the following modifications:

- Remove "compact" and "interior".
- Consider epigraphs, i. e. $F_t(K) = \left\{\left(\hat{x}, x^{n+1}\right) \in \mathbb{R}^{n+1} : x^{n+1} \geq u(\hat{x}, t)\right\}$.
- Observe that the semi-group property, §2.1 (4), follows as solutions to (1.1) for smooth initial data, which we have for positive times, with uniformly bounded gradient are unique.
- Property §2.1 (6) is fulfilled if we construct solutions as in [10]. Observe in particular that $u \leq v$ is preserved under mollifications.

With these modifications, [18, Theorem 3.1] extends to our situation, i. e. $U(\cdot, t_1) \leq U(\cdot, t_2)$ for $0 \leq t_1 \leq t_2$. Hence $\dot{U}(\cdot, t) \geq 0$. As $\dot{U}(0, t) > 0$ for $t > 0$, the strong maximum principle implies that $\dot{U}(\cdot, t) > 0$. □

The difference between homothetic solutions starting at different times tends to zero for large times.

Lemma 2.4. *Let $k : \mathbb{R}^n \to \mathbb{R}$ be continuous and positive homogeneous of degree one. Let $U \in C^\infty_{loc}(\mathbb{R}^n \times (0, \infty)) \cap C^0_{loc}(\mathbb{R}^n \times [0, \infty))$ be the homothetically expanding solution to (1.1) with $U(\cdot, 0) = k$. Let $T > 0$. Then*

$$U(\cdot, t + T) - U(\cdot, t) \to 0 \quad \text{as} \quad t \to \infty,$$

uniformly in C^k for any $k \in \mathbb{N}$.

Proof. We will prove that

$$\|U(\cdot, t + T) - U(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \to 0 \quad \text{as} \quad t \to \infty.$$

Then uniform gradient estimates and local higher derivative estimates, see [10, Theorems 2.3, 3.1 and 3.4], imply the claimed convergence.

According to [10, Theorem 3.1] and [17, Corollary 1], we deduce that $|H[U(\cdot, 1)]| \leq c$. Hence (1.2) implies $|H[U(\cdot, t)]| \leq \frac{c}{\sqrt{t}}$ and, as DU is uniformly bounded for $t \geq 1$, $\left|\dot{U}(\cdot, t)\right| \leq \frac{c}{\sqrt{t}}$. We integrate from t to $T+t$ and obtain the claimed convergence. □

As a consequence, we can prove Theorem 1.3 if (the graphs of) u_0 and U lie on the same side of k.

Proof of Theorem 1.3, 1st part: We will prove Theorem 1.3 under three additional assumptions:

(i) We have $U(\cdot, t) > k$ for any $t > 0$. (This means that k is singular at the origin, see Corollary 2.2. We consider linear functions k in Appendix A.)
(ii) For every $\delta > 0$, there exists $t_\delta > 0$ such that $u(\cdot, t_\delta) \geq k - \delta$.
(iii) $n \geq 3$.

Let $\delta > 0$ and $\varepsilon = 2\delta$. We obtain

$$u(\cdot, 0 + t_\delta) - (U(\cdot, 0) - \varepsilon) = u(\cdot, 0 + t_\delta) - k + 2\delta \geq \delta > 0 \tag{2.1}$$

on $\mathbb{R}^n$. The functions u and $U-\varepsilon$ evolve by (1.1). Using small spheres as barriers, the compact maximum principle implies that $u(\cdot,t+t_\delta)-(U(\cdot,t)-\varepsilon)>0$ for some time interval $[0,\zeta]$, $\zeta>0$. By comparison with large spheres and the compact maximum principle, we see that u and $U-\varepsilon$ grow at most polynomially at spatial infinity. On any bounded time interval of the form $[\zeta,T]$, the interior estimates of [10] imply uniform gradient bounds for $U(\cdot,t)-\varepsilon$. Hence the comparison principle of G. Barles, S. Biton, M. Bourgoing and O. Ley, see [1] or [6, Theorem A1], is applicable and implies that $u(\cdot,t+t_\delta)-(U(\cdot,t)-\varepsilon)\ge 0$.

On the other hand, we have $u_0\le k+\varepsilon$ in $\mathbb{R}^n\setminus B_r(0)$ for r sufficiently large. According to (1.2), we find $T>0$ such that $U(\cdot,T)\ge u_0$ in $B_r(0)$. Hence (1.3) implies that $U(\cdot,T)+\varepsilon\ge u_0$ in $\mathbb{R}^n$. As above, the maximum principle implies that $U(x,t+T)+\varepsilon\ge u(x,t)$ for all (x,t), $t\ge 0$.

Combining the above estimates, we get

$$U(\cdot,t)-\varepsilon\le u(\cdot,t+t_\delta)\le U(\cdot,t+t_\delta+T)+\varepsilon.$$

Lemma 2.4 implies that

$$|u(\cdot,t)-U(\cdot,t)|\le 3\varepsilon$$

for t sufficiently large. As $\varepsilon>0$ was arbitrary, we obtain C^0-convergence. According to [10], higher derivatives are uniformly bounded. Hence interpolation inequalities imply the claimed convergence. □

3. Barrier Construction

Let us describe the idea of the barrier construction in the case of Theorem 1.1. The proof of the corresponding statement for Theorem 1.3 is more complicated. We give it below.

Assume that k is as in Theorem 1.1. Consider $w:=k-|x|^{-\alpha}$ for $\alpha>0$. We wish to show that $H[w](x)>0$ for $|x|$ sufficiently large. Consider $H[w]\sqrt{1+|Dw|^2}$. According to the scaling behavior of the mean curvature of cones, there exists some $\varepsilon>0$ such that $H[k](x)\sqrt{1+|Dk|^2}\ge\varepsilon|x|^{-1}$. A direct calculations using the fundamental theorem of calculus, however, shows that $|H[w]\sqrt{1+|Dw|^2}-H[k]\sqrt{1+|Dk|^2}|\le c|x|^{-\alpha-2}$. Hence we obtain $H[w](x)\ge 0$ for $|x|$ sufficiently large as claimed.

In the case of Theorem 1.3, we obtain the barrier via a flow equation.

Lemma 3.1. *Let k be as in Theorem 1.3 and $n\ge 3$. Let X_0 denote the embedding vector of* graph k *outside the origin and ν its downwards pointing unit normal. Deform the embedding vector X according to*

$$\frac{d}{dt}X=-F\nu,\quad X(\cdot,0)=X_0,\quad F(X)=-\left(|X|^2\right)^{-\alpha}\equiv-|X|^{-\frac{n-2}{2}}.$$

Then for r sufficiently large, the image of $X(\cdot,1)$ in $(\mathbb{R}^n\setminus B_r(0))\times\mathbb{R}$ can be written as graph b, *$b:\mathbb{R}^n\setminus B_r(0)$, where b fulfills*

(a) $b\in C^\infty_{loc}$,
(b) $\sup\limits_{\mathbb{R}^n\setminus B_R(0)}|b-k|\to 0$ *as* $R\to\infty$,
(c) $b<k$,

(d) $H[b] > 0$.

Proof. The evolution equation is a first order partial differential equation. Standard results (for example, iterated application of the results in [11, §3.2.4]) and the evolution equations below ensure existence of a solution on $(\mathbb{R}^n \setminus B_R(0)) \times [0,1]$ for some large R. Hence b is smooth, if we can write $X(\cdot,1)$ as a graph. Standard methods (see, for example, [12, 13, 16]; we also use the notation used there) yield the following evolution equations

$$\begin{aligned}
\frac{d}{dt}g_{ij} =& -2Fh_{ij},\\
\frac{d}{dt}h_{ij} =& F_{;ij} - Fh_i^k h_{kj}\\
=& 4\alpha(\alpha+1)F|X|^{-4}\langle X, X_i\rangle\langle X, X_j\rangle - 2\alpha F|X|^{-2}(g_{ij} - \langle X,\nu\rangle h_{ij})\\
& - Fh_i^k h_{kj},\\
\frac{d}{dt}H =& \frac{d}{dt}\left(g^{ij}h_{ij}\right) = -g^{ik}g^{jl}h_{ij}\frac{d}{dt}g_{kl} + g^{ij}\frac{d}{dt}h_{ij}\\
=& (-F)\left(-|A|^2 - 4\alpha(\alpha+1)|X|^{-4}\left(|X|^2 - \langle X,\nu\rangle^2\right)\right)\\
& + (-F)\left(2\alpha|X|^{-2}(n - \langle X,\nu\rangle H)\right),\\
\frac{d}{dt}|A|^2 =& \frac{d}{dt}\left(g^{ij}h_{jk}g^{kl}h_{li}\right) = -2g^{ir}g^{js}h_{jk}g^{kl}h_{li}\frac{d}{dt}g_{rs} + 2g^{ij}h_{jk}g^{kl}\frac{d}{dt}h_{li}\\
=& 2F\,\mathrm{tr}\,A^3 - 8\alpha(\alpha+1)(-F)|X|^{-4}\langle X, X_i\rangle h^{ij}\langle X_j, X\rangle\\
& + 4\alpha(-F)|X|^{-2}\left(H - \langle X,\nu\rangle|A|^2\right),\\
\frac{d}{dt}\nu^\beta =& F_i g^{ij}X_j^\beta\\
=& 2\alpha|X|^{-2\alpha-2}\left(X^\beta - \langle X,\nu\rangle\nu^\beta\right).
\end{aligned}$$

We have the following geometric scaling: $|A|[k](p) \sim |p|^{-2}$ and $H[k](p) \sim |p|^{-1}$. Initially, we have $|A|^2(p) \le c_A|p|^{-2}$ for some constant $c_A > 0$. As long as $|A|^2(p) \le 2c_A|p|^{-2}$, we obtain

$$\begin{aligned}
\left|\frac{d}{dt}|A|^2\right| \le& c|F||A|^3 + c|F||X|^{-2}|A| + c|F||X|^{-2}\left(|A| + |X||A|^2\right)\\
\le& c|F||A|^3 + c|F||X|^{-2}|A| \qquad (3.1)\\
\le& c|F||X|^{-3} \le \frac{c}{|X|}|X|^{-2}. \qquad (3.2)
\end{aligned}$$

For the rest of the proof, we will always assume that r is sufficiently large, i. e. our conclusions hold in $(\mathbb{R}^n \setminus B_r(0)) \times \mathbb{R}$. The evolution equation above justifies our assumption $|A|^2(p) \le 2c_A|p|^{-2}$ for $t \le 1$; hence we will assume $t \in [0,1]$.

The stability condition imposed on k (Condition (v) of Definition 1.2) ensures that there exists a neighbourhood $\mathcal{N} \subset \mathbb{R}^{n+1}$ of the set on which $H = 0$ which is invariant under homotheties and translations parallel to e_{n+1}, and an $\varepsilon > 0$, such that $|p|^2|A|^2(p) \le \left(\frac{n-2}{2}\right)^2 - \varepsilon$ on $\mathcal{N}$. Now (3.2) ensures that $|p|^2|A|^2(p) < \left(\frac{n-2}{2}\right)^2$ holds on $\mathcal{N} \cap ((\mathbb{R}^n \setminus B_r(0)) \times \mathbb{R})$ for r sufficiently large. In $\mathcal{N}$, the maximum principle

applied to

$$\begin{aligned}\frac{d}{dt}H \geq & (-F)\left(-|A|^2 + \left(\tfrac{n-2}{2}\right)^2 |X|^{-2} - 2\alpha|X|^{-2}\langle X,\nu\rangle H\right)\\ > & 2\alpha F|X|^{-2}\langle X,\nu\rangle H\end{aligned}$$

implies that $H > 0$ for $0 < t \leq 1$.

Let $\mathcal{N}^c := \left(\mathbb{R}^{n+1}\setminus\mathcal{N}\right)\setminus(B_r(0)\times\mathbb{R})$. There exists $\delta > 0$ such that $|p|\cdot H[k](p) \geq \delta$ in $\mathcal{N}^c$. Equation (3.1) implies that $\left|\frac{d}{dt}|A|\right| \leq \frac{c}{|X|}\frac{1}{|X|}$. Hence $H[k] > 0$ is preserved in $\mathcal{N}^c$ for $0 \leq t \leq 1$.

Note also that $\left|\frac{d}{dt}X\right|$ becomes small near infinity. Hence $|X|$ hardly changes during the evolution. The evolution equation for ν implies that the normal hardly changes during the evolution. Hence $X(\cdot,1)$ can be written as graph of a smooth function. Moreover $b < k$ as ν hardly changes. Bounds on $k-b$ are immediate from the decay of $|F|$. The Lemma follows. □

Lemma 3.2. *Let k be as in Lemma 3.1. Let*

$$u \in C^\infty_{loc}\left(\mathbb{R}^n\times(0,\infty)\right)\cap C^0_{loc}\left(\mathbb{R}^n\times[0,\infty)\right)$$

be a solution to (1.1) *such that $u_0 := u(\cdot,0)$ fulfills* (1.3). *Let $\delta > 0$. Then there exists $t_\delta > 0$ such that*

$$u(\cdot,t_\delta) \geq k-\delta.$$

Proof. Fix $R > 0$ such that $|u_0 - k(x)| < \frac{\delta}{2}$ for $|x| \geq R$. Let $m := -\inf(u_0-k)$. We may assume that $m > \delta$. Let b be the barrier obtained in Lemma 3.1. Assume that b is defined in $\mathbb{R}^n\setminus B_{R_1}(0)$ and set $m_1 := -\inf(b-k)$. Observe that for $\lambda > 0$

$$b^\lambda(x) = \lambda b\left(\frac{x}{\lambda}\right)$$

also fulfills the conditions on b in Lemma 3.1. Choose $\lambda > 0$ such that $\lambda m_1 > m$ and $\lambda R_1 > R$. Hence $B(x,t) := \max\left\{U(x,t)-m, b^\lambda(x)-\frac{\delta}{2}\right\}$ (and $B(x,t) := U(x,t)-m$, where b^λ is not defined) is a subsolution to (1.1) in the viscosity sense. As $B(\cdot,t)$ is asymptotic to a cone, the maximum principle [1, Theorem 2.1] implies that $u(\cdot,t) \geq B(\cdot,t)$ for all $t \geq 0$. There exists $r > 0$ such that $b \geq k-\delta$ on $\mathbb{R}^n\setminus B_r(0)$. Positivity of $U(\cdot,t)-k$ for any $t > 0$ and (1.2) imply that there exists $t_\delta > 0$ such that $U(x,t)-m \geq k(x)$ for $|x| \leq r$ and all $t \geq t_\delta$. We obtain $u(\cdot,t) \geq k-\delta$ for all $t \geq t_\delta$. □

Proof of Theorem 1.3, 2nd part: Here we will prove Theorem 1.3 under the assumption that k is not a linear function, see Assumption (i) in the first part of the proof. This case is considered in Appendix A. We will also assume that $n \geq 3$. If $n = 1$ or $n = 2$, k is convex. We will consider this case independently.

According to Corollary 2.2, $U(\cdot,t)-k > 0$ for $t > 0$. Lemma 3.2 implies that t_δ as in Assumption (ii) in the first part of the proof of Theorem 1.3 exists. Hence the result follows from the proof given there. □

The proof becomes simpler if the function k is convex.

Proof of Theorem 1.3, 3rd part: Here we assume that k is convex. This is obvious if $n=1$. If $n=2$, we observe, that outside the origin, one principal curvature of a cone vanishes. Hence $H \ge 0$ is equivalent to local convexity outside the origin. As k is a cone, by continuity, the function k is convex on all of $\mathbb{R}^n$.

For any point in graph k, there exists a supporting hyperplane, which we assume to be graph l for some linear function l. This hyperplane is a supporting hyperplane for a half-line in graph k. Hence for every $\delta > 0$ the considerations in Appendix A imply the existence of $t_\delta > 0$ such that $u(\cdot, t) \ge l - \delta$. As the decay assumption (1.3) is independent of the direction in which we approach infinity, the results of Appendix A and considerations as in the proof in Appendix B, applied to the results in Appendix A after an appropriate rotation, imply that $t_\delta > 0$ can be chosen independently of l. Hence Assumption (ii) in the first part of the proof of Theorem 1.3 is fulfilled. We can now proceed as in the first part of the proof. □

Remark 3.3. If $k \le u_0 \le U(\cdot, T)$ for some $T > 0$, then the proof of Theorem 1.3 implies the decay rate

$$\sup_{\mathbb{R}^n} |u(\cdot, t) - U(\cdot, t)| \le \tfrac{c}{\sqrt{t}}.$$

According to the considerations in the proof of Lemma 2.4, this rate is sharp.

4. BV-Perturbations

In this section we discuss how to replace (1.3) by a weaker condition that allows for additional decaying perturbations in BV_{loc} in the case $n \ge 2$.

In the following, it is possible to consider $u \in C^0_{loc}(\mathbb{R}^n) \cap BV_{loc}(\mathbb{R}^n)$. For the sake of an easier presentation, however, we will assume that $u \in C^1_{loc}(\mathbb{R}^n)$. Set $\|u\|_{BV(\Omega)} := \int\limits_\Omega |u| + |Du|$. We generalize (1.3) as follows: Assume that there exists $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$, depending on u, such that $\varepsilon(r) \to 0$ as $r \to \infty$ and for all $r > 0$

$$\sup_{x \in \mathbb{R}^n \setminus B_r(0)} \|u_0 - k\|_{BV(B_1(x) \cap \{|u_0 - k| > \varepsilon(r)\})} < \varepsilon(r). \tag{4.1}$$

In order to show that our results remain valid under this initial assumption, it suffices to prove that (4.1) implies a condition of the form (1.3) for some positive time. More precisely, it suffices to show that for every $\delta > 0$ there exist $r > 0$ and $t_0 = t_0 > 0$, both depending on u, such that

$$|u(\cdot, t_0) - k| < \delta \quad \text{in} \quad \mathbb{R}^n \setminus B_r(0). \tag{4.2}$$

This is a consequence of Brakke's clearing out lemma [3]:

Lemma 4.1. *Let $k : \mathbb{R}^n \to \mathbb{R}$ be a cone which is smooth outside the origin. Let $u_0 \in C^1_{loc}(\mathbb{R}^n)$. Assume that there exists $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that u_0 fulfills (4.1). Let $u \in C^\infty_{loc}(\mathbb{R}^n \times (0, \infty)) \cap C^0_{loc}(\mathbb{R}^n \times [0, \infty))$ be a solution to (1.1) with $u(\cdot, t) = u_0$. Let $\delta > 0$. Then there exist $t_0 > 0$ and $r > 0$ such that (4.2) is fulfilled.*

Proof. Fix $x \in \mathbb{R}^n$ and $0 < \rho < 1$. Let us assume that $\sup |Dk| \le G$ for some $G \ge 1$. Then we get $k(y) \le k(x) + \rho G$ for $y \in B_\rho(x)$. Hence the ball $B_\rho(x, k(x) + 2\rho + \rho G) \subset$

$\mathbb{R}^{n+1}$ lies above $k+\rho$. Let us estimate the area A of graph u_0 above $k+\rho$ in $B_\rho(x)\times\mathbb{R}$. Set $\Omega := \{y \in B_\rho(x) : u_0(y) > k(y) + \rho\}$. We have

$$\begin{aligned} A = \int\limits_{\Omega} \sqrt{1+|Du_0|^2} &\le \int\limits_{\Omega\cap\{|Du_0|\ge 2G\}} 2|Du_0| + \int\limits_{\Omega\cap\{|Du_0|<2G\}} \sqrt{1+4G^2} \\ &\le \int\limits_{\Omega\cap\{|Du_0|\ge 2G\}} 4|D(u_0-k)| + \int\limits_{\Omega\cap\{|Du_0|<2G\}} 3G\frac{|u_0-k|}{\rho} \\ &\le \left(4+\tfrac{3G}{\rho}\right)\|u_0-k\|_{BV(\Omega)} \le \left(4+\tfrac{3G}{\rho}\right)\|u_0-k\|_{BV(B_1(x)\cap\{|u_0-k|>\varepsilon(|x|)\})} \end{aligned}$$

for $\varepsilon = \varepsilon(|x|) < \rho$. According to (4.1), the right-hand side is small for all $|x|$ sufficiently large. Hence the clearing out lemma, [3, Lemma 6.3] or [9, Proposition 4.23], implies that $(x, k(x) + 2\rho + \rho G) \notin \operatorname{graph} u(\cdot, t_0)$ for some $t_0 = c\rho^2 > 0$. A similar argument ensures that $\{x\}\times((-\infty, k(x)-2\rho-\rho G)\cup(k(x)+2\rho+\rho G,\infty))\cap \operatorname{graph} u(\cdot,t_0) = \emptyset$ for all $|x|$ sufficiently large. The claim follows. □

All the arguments in the proof of Theorem 1.3 extend to such perturbations u if we use (4.2) instead of (1.3).

Observe also that instead of (4.1) we could require directly that the area A of graph u_0 estimated in the proof of Lemma 4.1 is small enough to apply the clearing out lemma.

Appendix A. Stability of $\mathbb{R}^n$ under Mean Curvature Flow

Remark A.1. In this section, we will always assume that our solutions to mean curvature flow are graphical. For other complete solutions with corresponding behavior at infinity, such theorems are also true if the solutions exist for $t \in [0,\infty)$. A graphical solution, which is initially above the considered solution, shifted by ε, can be used as a barrier. This implies these more general results.

In the following theorem, we show that a solution to (1.1) leaves every half-space if near infinity it is initially not too far inside the half-space. The following proof is due to G. Huisken.

Theorem A.2. *Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be continuous and assume that for every $\varepsilon > 0$ there exists $r > 0$ such that $u_0 < \varepsilon$ in $\mathbb{R}^n \setminus B_r(0)$. Let $u \in C^\infty_{loc}(\mathbb{R}^n \times (0,\infty)) \cap C^0_{loc}(\mathbb{R}^n \times [0,\infty))$ be a solution to (1.1). Then*

$$\limsup_{t\to\infty} \sup_{\mathbb{R}^n} u(\cdot,t) \le 0.$$

Proof. Consider $\Phi(x,t) := \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$. Set $M_t := \operatorname{graph} u(\cdot,t)$. We claim that

$$\frac{d}{dt}\Phi(X,t) - \Delta^{M_t}\Phi(X,t) = \frac{d}{dt}\Phi(X,t) - g^{ij}(\Phi(X,t))_{;ij} \ge 0.$$

This is equivalent to

$$\frac{d}{dt}\Psi(X,t) - g^{ij}(\Psi(X,t))_{;ij} - g^{ij}(\Psi(X,t))_i(\Psi(X,t))_j \ge 0,$$

where

$$\Psi = \Psi(X,t) = \log \Phi(X,t) + \frac{n}{2}\log(4\pi) = -\frac{n}{2}\log t - \frac{|X|^2}{4t}.$$

We obtain

$$\begin{aligned}
\frac{d}{dt}\Psi(X,t) &= -\frac{n}{2t} + \frac{|X|^2}{4t^2} + \frac{\langle X,\nu\rangle H}{2t},\\
\Psi_i &= -\frac{X^\alpha \overline{g}_{\alpha\beta} X_i^\beta}{2t},\\
\Psi_{;ij} &= -\frac{g_{ij}}{2t} + \frac{\langle X,\nu\rangle h_{ij}}{2t},\\
\frac{d}{dt}\Psi - g^{ij}\Psi_{;ij} - g^{ij}\Psi_i\Psi_j &= \frac{|X|^2}{4t^2} - \frac{X^\alpha \overline{g}_{\alpha\beta} X_i^\beta g^{ij} X_j^\gamma \overline{g}_{\gamma\delta} X^\delta}{4t^2}\\
&= \frac{1}{4t^2}\left(|X|^2 - X^\alpha \overline{g}_{\alpha\beta}\left(\overline{g}^{\beta\gamma} - \nu^\beta\nu^\gamma\right)\overline{g}_{\gamma\delta}X^\delta\right)\\
&= \frac{\langle X,\nu\rangle^2}{4t^2} \ge 0.
\end{aligned}$$

Define $\eta := (0,\ldots,0,1)$. Then $u^X := \eta_\alpha X^\alpha$ equals u up to a change in the parametrization. We get

$$\frac{d}{dt}u^X - g^{ij}(u^X)_{;ij} = \eta_\alpha(-H\nu^\alpha + g^{ij}h_{ij}\nu^\alpha) = 0.$$

Using large balls as barriers, we see that for every $T > 0$ and every $\varepsilon > 0$, there exists $r > 0$ such that $u^X \le \varepsilon/2$ on $\operatorname{graph} u(\cdot,t)|_{\mathbb{R}^n\setminus B_r(0)}$, $t \in [0,T]$. Fix $t_0 >$ small and $\varepsilon > 0$. As $\Phi(\cdot,t)$ is a continuous positive function, there exists $a > 0$ such that $a\Phi(X,t_0) + \varepsilon - u^X(\cdot,t_0) \ge 0$ in $B_r(0)\times\mathbb{R}$. The considerations above ensure that $a\Phi(X,t) + \varepsilon - u^X(X,t) \ge \varepsilon/2$ for t in a bounded time interval and $X \in (\mathbb{R}^n \setminus B_r(0))\times\mathbb{R}$ for r sufficiently large, depending in particular on the time interval considered. As

$$\frac{d}{dt}\left(a\Phi - u^X\right) - g^{ij}\left(a\Phi - u^X\right)_{;ij} \ge 0,$$

the maximum principle implies that $a\Phi + \varepsilon - u^X \ge 0$ for all $t \ge t_0$. As $a\Phi \to 0$, uniformly as $t\to\infty$, we deduce that

$$\limsup_{t\to\infty}\sup_{x\in\mathbb{R}^n} u(x,t) \le 2\varepsilon.$$

The claim follows. □

As a corollary to Theorem A.2 we obtain the following stability theorem, which generalizes [6, Appendix C] to all dimensions.

Theorem A.3. *Let $u_0 \in C^0_{loc}(\mathbb{R}^n)$ fulfill (1.3). Let $u \in C^\infty_{loc}(\mathbb{R}^n\times(0,\infty)) \cap C^0_{loc}(\mathbb{R}^n\times[0,\infty))$ be a solution to (1.1). Then*

$$\sup_{\mathbb{R}^n}|u(\cdot,t)| \to 0 \quad \text{as} \quad t\to\infty.$$

Moreover,

$$\|D^\alpha u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \to 0 \quad \text{as} \quad t\to\infty$$

for every derivative $D^\alpha u$.

Proof. Theorem A.2 yields the upper bound; the lower bound follows similarly. This implies C^0-convergence. Uniform interior derivative estimates and interpolation inequalities yield C^k-convergence for any $k \in \mathbb{N}$. □

Appendix B. Uniform Stability for Families of Solutions

Our stability result extends to families of solutions as follows.

Theorem B.1. *Let k and U be as in Theorem 1.3. Let $(u^i)_i$ be a family of functions such that each function fulfills the conditions on u in Theorem 1.3. If* (1.3) *is uniformly fulfilled in the sense that*

$$\sup_i \sup_{\mathbb{R}^n \setminus B_r(0)} |u^i(\cdot, 0) - k| \to 0 \quad \text{as} \quad r \to \infty,$$

then

$$\sup_i \left\| D^\alpha (u^i - U)(\cdot, t) \right\|_{L^\infty(\mathbb{R}^n)} \to 0 \quad \text{as} \quad t \to \infty,$$

for any derivative D^α.

Proof. Fix $\varepsilon > 0$. Use $u^+ + \varepsilon$ and $u^- - \varepsilon$ as barriers for u^i, where $u^\pm$ are solutions to (1.1) such that $u^+(\cdot, 0) \ge u^i(\cdot, 0) \ge u^-(\cdot, 0)$ and $u^\pm(\cdot, 0)$ fulfill the assumptions on u_0 of Theorem 1.3. □

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